# Combinatorial Games

Lecture 11   Mar 21, 2021

# Copycat and pairing strategies

- Players A (Alice) and B (Bob) are playing a game with A doing the first move

**Scenario 1:** B somehow copies the move of A so that he always has a move after A, so he has a winning strategy!

**Scenario 2:** A makes a special first move, after that, A somehow copies the move of B so that she always has a move after A, so she a winning strategy!

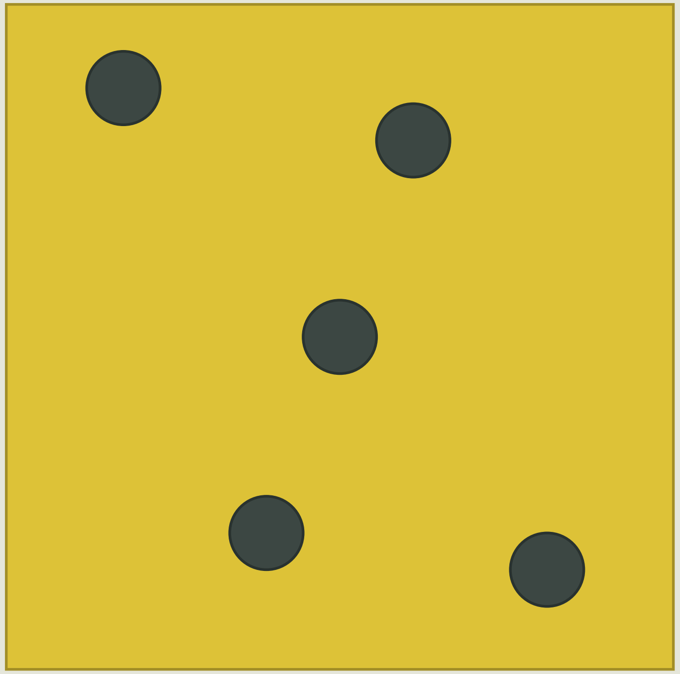
**Winning strategy:** A way of winning the game regardless of how the other player plays the game.

**Warning:** If you are Alice or Bob, you can not decide what the **best move** for the other person is!

# Example*

- Q1. A and B take turns placing identical circular coins on a (large) square table.

The person who can not place a coin loses the game. Which person has a winning strategy?

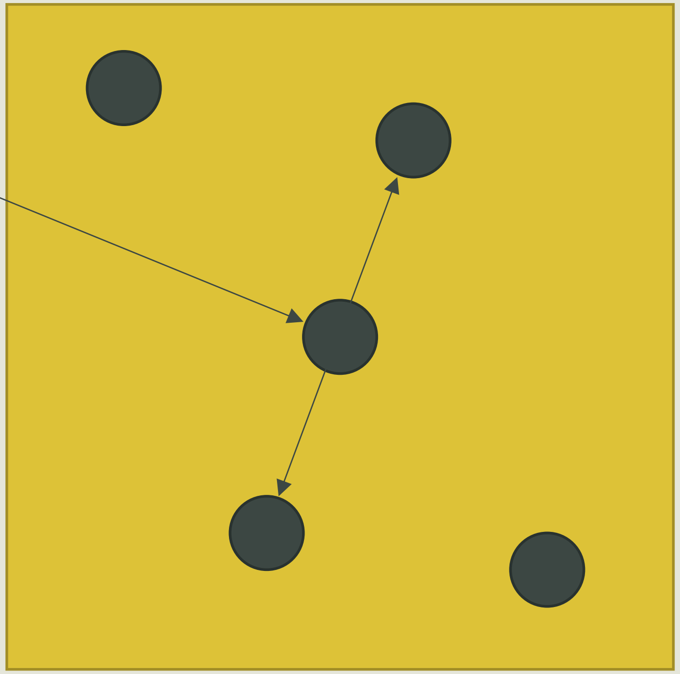


* following Olympiad Combinatorics by Pranav A. Sriram (August 2014)

# Solution

A has a winning strategy:

- For her first move she must put a coin at the center
- For her subsequent moves she must put a coin at the opposite location of Bob's coin



# Example*

- Q2. $N = p_1 \times p_2 \times \cdots \times p_k$ where each $p_i$ is a prime number and $k \geq 3$.

A and B take turns to write composite divisors of N according to the following rule:

- One may not write N
- There may never appear to coprime numbers
- There may never appear two numbers one of which divides the other.

The last person who can write a number wins. Who has a winning strategy?

Example: $N = 2 \times 3 \times 5 \times 7 \times 11$

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# Solution

- A has a winning strategy:

- For her first move she writes $p_1 \, p_2$

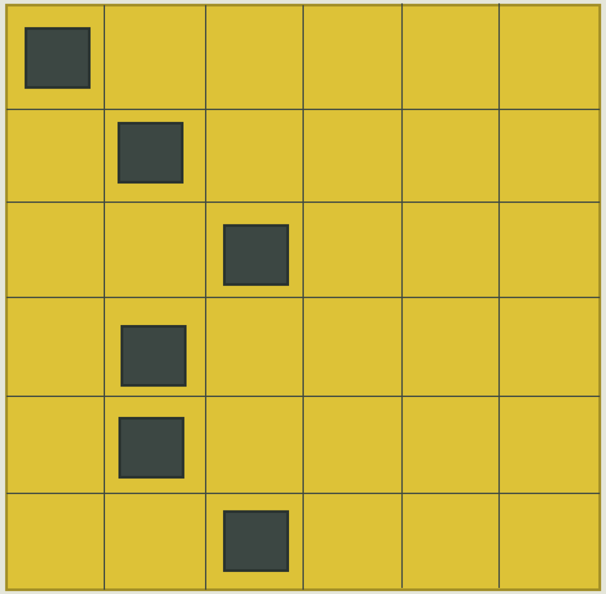
# Example*

- Q3. Alice and Bob play a game on a 6x6 table. On his or her turn, a player writes a new number in an empty square. At the end:

In each row, the square with the greatest number is colored black.

Alice wins if black squares include a path from top to bottom.

Bob wins otherwise.



Who has a winning strategy?

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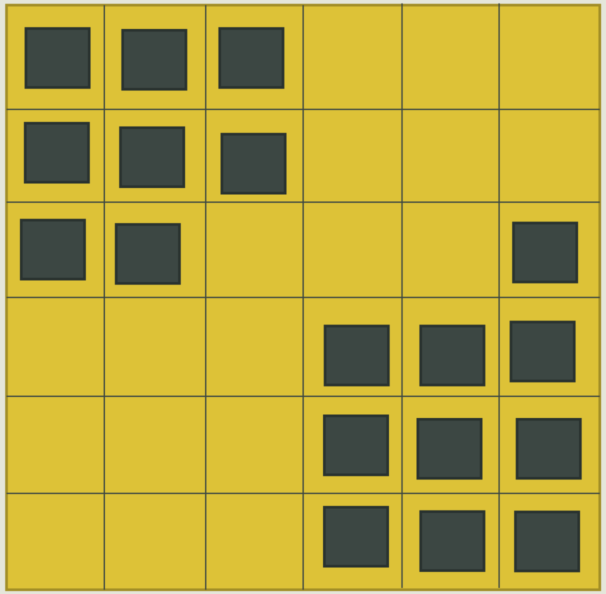
# Solution

BOB has a winning strategy:

Bob can play in such a way that the greatest number in each row happens at one of the black square in the following picture



# Example*

- Q4. On a 5x5 board, A and B take turns to mark the squares with X and O (as in Tic Tac Toe).  Alice wins if she can make one full row, column, or diagonal. Can B prevent A from winning?

- Hint: This is in some way similar to Q1.

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# Solution

| | | | | |
|---|---|---|---|---|
| **1** | **7** | **4** | **7** | **2** |
| **5** | **9** | **9** | **10** | **6** |
| **3** | **12** | | **10** | **3** |
| **5** | **12** | **11** | **11** | **6** |
| **2** | **8** | **4** | **8** | **1** |

There is an identical pair of numbers
In each row, column, and diagonal

There are 5 rows, 5 columns,
and 2 diagonals = 12 winning paths for A
That we must block

# Example*

- Q5. A and B play on a 5x5 board by putting 1 and 0 in each square, in their turn.

At the end they calculate the sum of numbers in each of nine 3x3 sub-squares.

Alice's score is the largest of these 9 numbers.

What is the largest number Alice can get regardless of Bob's play.

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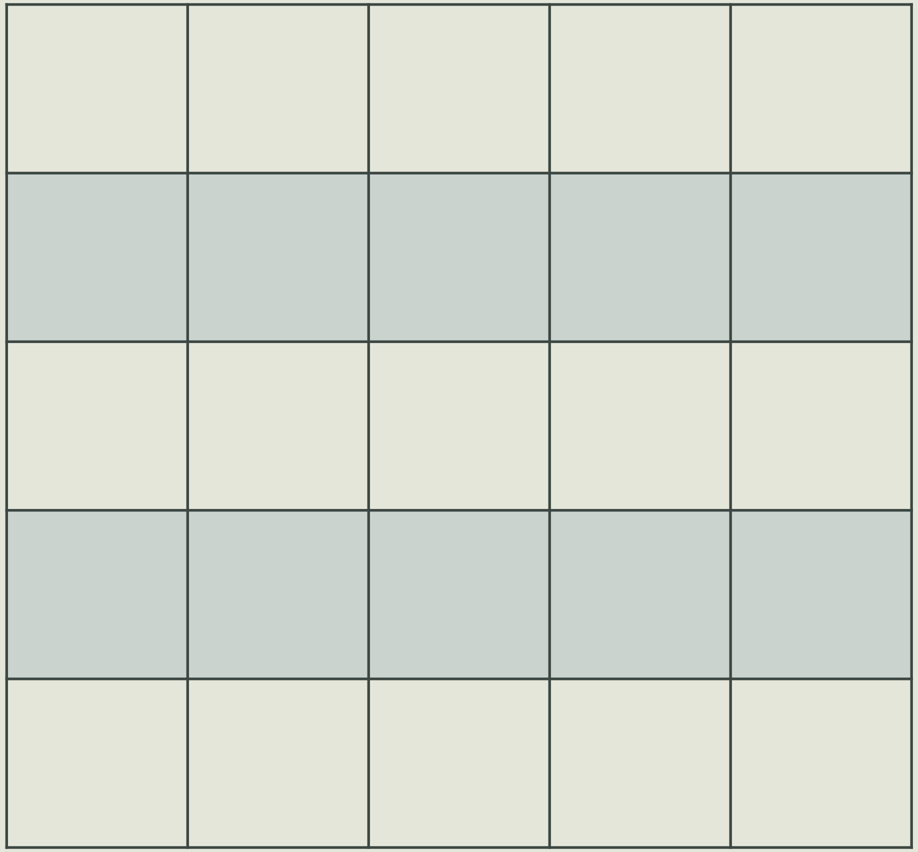
# Solution

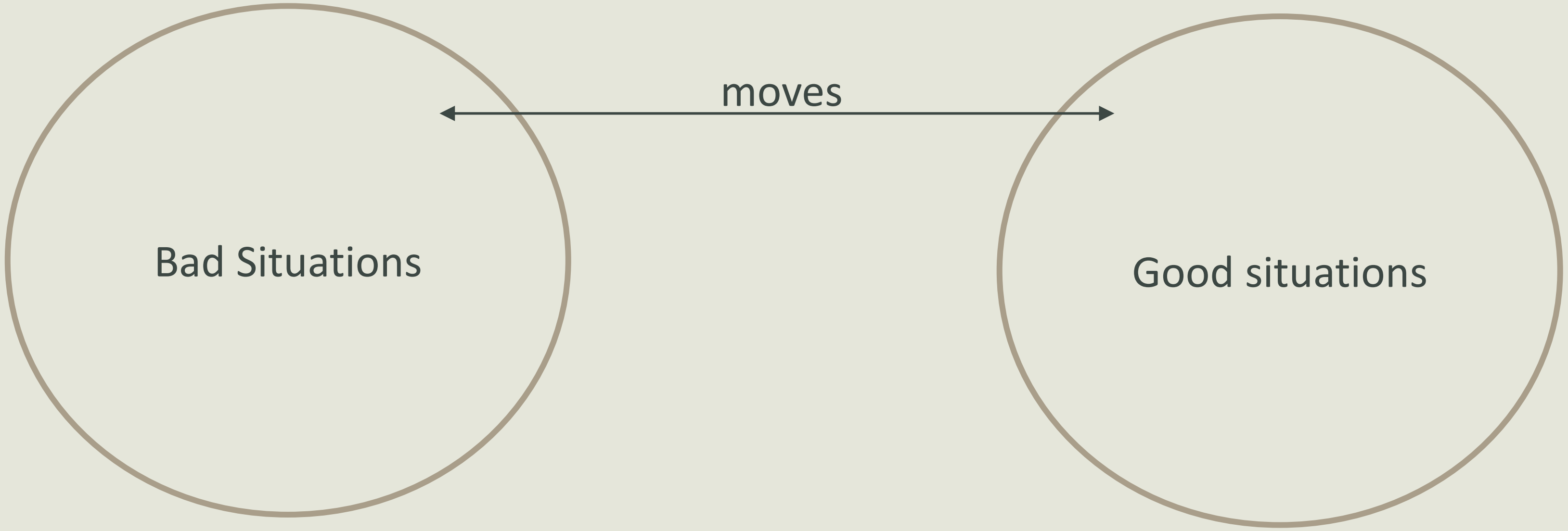


# Positional Analysis techniques

Divide the possible positions into two groups: **Good** and **Bad**

So that one can only move from bad to good and vice versa

Then depending on whether the initial position is bad or good, the person in good position should play correctly to preserve his/her position



# Example*

- Q6. A and B alternatively write S and O on a 1x2000 grid. The first who completes a 1x3 block reading SOS wins.

Does anyone has a winning strategy?

Comment: The number 2000 is not important; for any large enough even number the problem should have an identical solution.

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# Solution

- Bob has a winning strategy

- Observation: B must never write anything between two S that are two empty spaces apart:  S __ __ S .
- He can create S __ __ S in two steps (to avoid draw), but then avoid filling in these two spots or any such pairs before Alice. Note that in B’s turn, always an odd number of empty squares is left. So he does not have to fill any of such pairs.

# Example*

- Q7. **Alice** and **Bob** play the following game.

First A writes one of 1, 2, 3

Then B chooses a number and adds it to number A has chosen

This repeats

So we get a sequence $a_1$ , $a_1 + a_2$, $a_1 + a_2 + a_3, \cdots$ where each $a_i$ is 1, 2, or 3.

The first one who gets 100 wins.

Who has a winning strategy?

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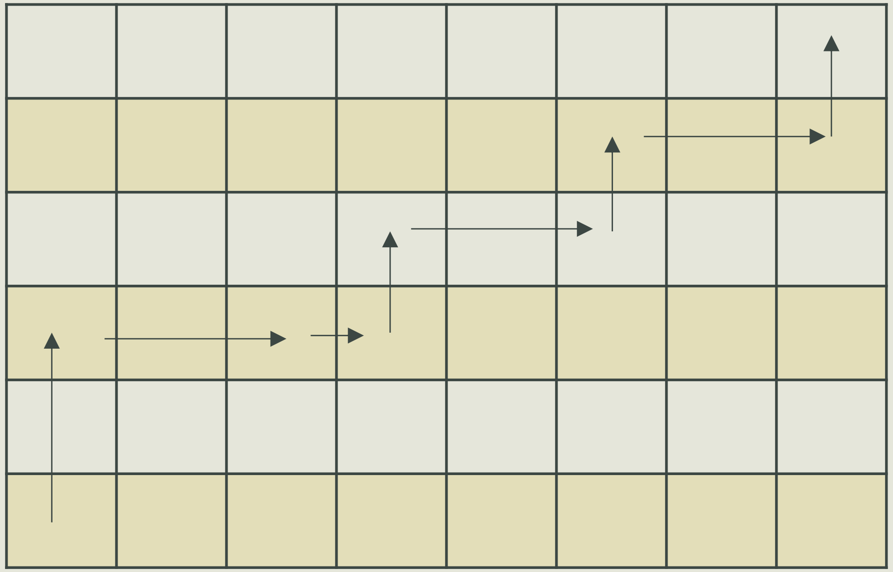
# Solution

- B has a  winning strategy: He can always choose $a_i \in \{1,2,3\}$ such that ends up with a. multiple of 4 (and, thus, keep A away from getting a sum that is a multiple of 4)

- <u>Observation</u>: If we replace 100 with any number which is not divisible by 4, then A has a winning strategy

# Example*

- Q8. In an $m \times n$ table, there is a piece in the lower leftmost square. A and B move the piece alternatively, by moving it right or up any number of squares. The player who reaches the opposite corner wins. For what values of $(m, n)$, A has a winning strategy?



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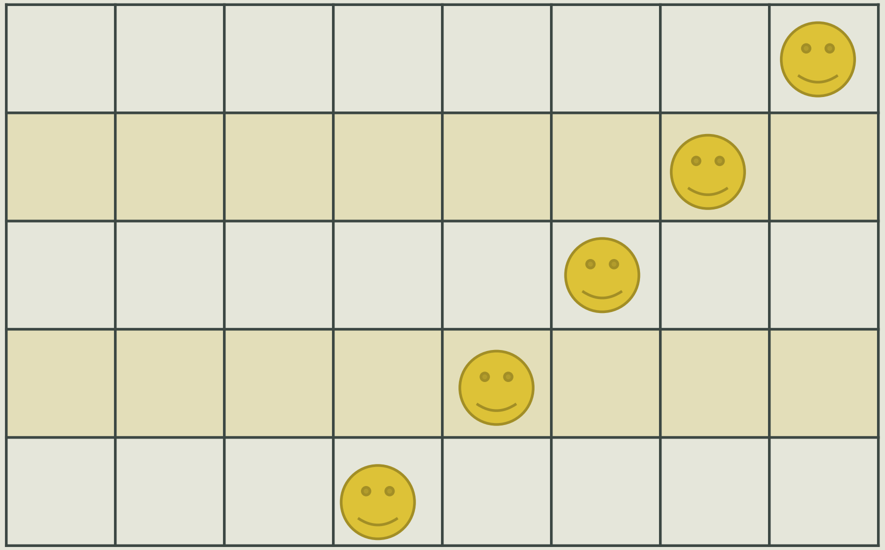
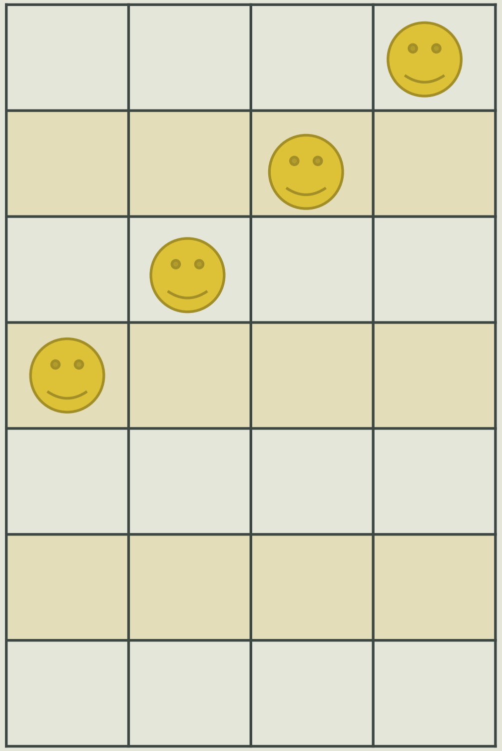
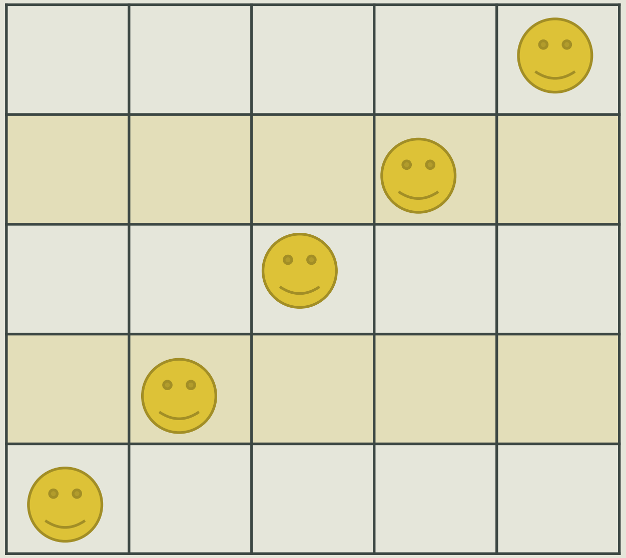
# Solution







# Example*

- Q8. Starting from the number $N$= 1,000,000, Alice and Bob, in each turn, replace N with $N-1$ or $\left\lfloor \frac{N+1}{2} \right\rfloor$. The first person who gets to 1 wins. Who has a winning strategy?

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